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UNIT-3 : Differentiation

Definition :-

1) Let $f: S \to R$ be a function and $c \in S$ then we say that f is derivative at c, if $\underset{x \to c}{Lt} \frac{f(x) - f(c)}{x - c}$ exists. It is denoted by $f'(c)$.

$$\therefore f'(c) = \underset{x \to c}{Lt} \frac{f(x) - f(c)}{x - c}$$

2) **Left hand derivative :**

Let $f: S \to R$ be a function and $c \in S$ then we say that f is left hand derivale at c, if $\underset{x \to c}{Lt} \frac{f(x) - f(c)}{x - c}$ exists. It is denoted by $Lf'(c)$ (or) LHD.

$$\therefore Lf'(c) \text{ (or) } L.H.D = \underset{x \to c^-}{Lt} \frac{f(x) - f(c)}{x - c}$$

3) **Right hand derivative :**

Let $f: S \to R$ be a function and $c \in S$ then we say that f is right hand derivative at c, if $\underset{x \to c^+}{Lt} \frac{f(x) - f(c)}{x - c}$ exists. It is denoted by $Rf'(c)$ (or) R.H.D.

$$\therefore RHD = \underset{x \to c^+}{Lt} \frac{f(x) - f(c)}{x - c}$$

NOTE :

1) $f'(c) = \underset{x \to c}{Lt} \frac{f(x) - f(c)}{x - c} = \underset{x \to c^-}{Lt} \frac{f(x) - f(c)}{x - c} = \underset{x \to c^+}{Lt} \frac{f(x) - f(c)}{x - c}$

1) S.T $f(x) = \begin{cases} 1 - x, & x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$ has no derivative at $x = 1$.

Sol: $f(x) = \begin{cases} 1 - x, & x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$

$LHD = \underset{x \to 1^-}{Lt} \frac{f(x) - f(c)}{x - c}$ $\qquad RHD = \underset{x \to 1^+}{Lt} \frac{f(x) - f(c)}{x - c}$

S.L.V.

$$= \underset{x\to 1^-}{Lt} \frac{f(x)-f(1)}{x-1} \qquad = \underset{x\to 1^+}{Lt} \frac{f(x)-f(1)}{x-1}$$

$$= \underset{x\to 1^-}{Lt} \frac{1-x-0}{x-1} \qquad = \underset{x\to 1^+}{Lt} \frac{x^2-1-0}{x-1}$$

$$= \underset{x\to 1^-}{Lt} \frac{-\cancel{(x-1)}}{\cancel{x-1}} \qquad = \underset{x\to 1^+}{Lt} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$= 1+1$$

$$= -1 \qquad = 2$$

$LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x=1$

2) prove that $f(x) = |x-1|$ has no derivative at $x=1$.

Sol: $f(x) = |x-1|$

$$LHD = \underset{x\to 1^-}{Lt} \frac{f(x)-f(c)}{x-c} \qquad RHD = \underset{x\to 1^+}{Lt} \frac{f(x)-f(c)}{x-c}$$

$$= \underset{x\to 1^-}{Lt} \frac{f(x)-f(1)}{x-1} \qquad = \underset{x\to 1^+}{Lt} \frac{f(x)-f(1)}{x-1}$$

$$= \underset{x\to 1^-}{Lt} \frac{-(x-1)-0}{x-1} \qquad = \underset{x\to 1^+}{Lt} \frac{x-1-0}{x-1}$$

$$= -1 \qquad = 1$$

$LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x=1$

3) S.T $f(x) = \begin{cases} 2-x, & x<0 \\ 2+x, & x\geq 0 \end{cases}$ has no derivative at $x=0$ (origin)

Sol: Given $f(x) = \begin{cases} 2-x, & x<0 \\ 2+x, & x\geq 0 \end{cases}$

$$LHD = \underset{x\to 0^-}{Lt} \frac{f(x)-f(c)}{x-c} \qquad RHD = \underset{x\to 0^+}{Lt} \frac{f(x)-f(c)}{x-c}$$

$$= \underset{x\to 0^-}{Lt} \frac{f(x)-f(0)}{x-0} \qquad = \underset{x\to 0^+}{Lt} \frac{f(x)-f(0)}{x-0}$$

$$= \underset{x\to 0^-}{Lt} \frac{\cancel{2}-x-\cancel{2}}{x} \qquad = \underset{x\to 0^+}{Lt} \frac{\cancel{2}+x-\cancel{2}}{x}$$

$$= \underset{x\to 0^+}{Lt} \frac{x}{x}$$

$$= -1 \qquad = 1$$

LHD $\neq$ RHD

$\therefore f(x)$ is has no derivative at $x=0$

4) S.T $f(x) = \begin{cases} 1+x, & x<2 \\ 5-x, & x \geq 2 \end{cases}$ has no derivative at $x=2$

Sol: Given $f(x) = \begin{cases} 1+x, & x<2 \\ 5-x, & x \geq 2 \end{cases}$

$$LHD = \underset{x \to 2^-}{Lt} \frac{f(x)-f(c)}{x-c}$$

$$= \underset{x \to 2^-}{Lt} \frac{f(x)-f(2)}{x-2}$$

$$= \underset{x \to 2^-}{Lt} \frac{1+x-3}{x-2}$$

$$= \underset{x \to 2^-}{Lt} \frac{x-2}{x-2}$$

$$RHD = \underset{x \to 2^+}{Lt} \frac{f(x)-f(c)}{x-c}$$

$$= \underset{x \to 2^+}{Lt} \frac{f(x)-f(2)}{x-2}$$

$$= \underset{x \to 2^+}{Lt} \frac{5-x-3}{x-2}$$

$$= \underset{x \to 2^+}{Lt} \frac{2-x}{x-2}$$

$$= \underset{x \to 2^+}{Lt} \frac{-\cancel{(x-2)}}{\cancel{x-2}}$$

$$= -1$$

RHD $\neq$ LHD

$\therefore f(x)$ has no derivative at $x=2$

5) S.T $f(x) = |x| + |x-1|$ has no derivative at $x=0, 1$.

Sol: Given $f(x) = |x| + |x-1|$

Derivative at $x=0$:

If $x<0, x<1$

then $|x| = -x$

$x<1 \Rightarrow (x-1)<0$

then $|x-1| = -(x-1)$

$f(x) = -x-x+1$

$= 1-2x$

If $x>0, x<1$

then $|x| = x$

$x<1 \Rightarrow (x-1)<0$

then $|x-1| = -(x-1)$

$f(x) = x-x+1$

$= 1$

Also $f(0) = |0| + |0-1|$

$= 1$

Now, $f(x) = |0| + |0-1|$

$= 1$

Now, $f(x) = \begin{cases} 1-2x, & x < 0 \\ 1, & x \geq 0 \end{cases}$

$LHD = \underset{x \to 0^-}{Lt} \dfrac{f(x) - f(0)}{x - 0}$

$= \underset{x \to 0^-}{Lt} \dfrac{(1-2x) - 1}{x}$

$= \underset{x \to 0^-}{Lt} \dfrac{-2\cancel{x}}{\cancel{x}}$

$= -2$

$RHD = \underset{x \to 0^+}{Lt} \dfrac{f(x) - f(0)}{x - 0}$

$= \underset{x \to 0^+}{Lt} \dfrac{1-1}{x}$

$= \dfrac{0}{x}$

$= 0$

$LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x = 0$

Derivative at $x = 1$:-

If $x > 0, x < 1$

Then $|x| = -x$

$x < 1 \Rightarrow (x-1) < 0$

Then $|x-1| = -(x-1)$

$f(x) = x - x + 1$

$= 1$

If $x > 0, x < 1$

Then $|x| = x$

$x > 1 \Rightarrow (x-1) > 0$

Then $|x-1| = (x-1)$

$f(x) = x + x - 1$

$= 2x - 1$

Also, $f(1) = |1| + |1-1|$

$= 1$

Now, $f(x) = \begin{cases} 1, & x < 1 \\ 2x-1, & x > 1 \end{cases}$

$LHD = \underset{x \to 1^-}{Lt} \dfrac{f(x) - f(1)}{x-1}$

$= \underset{x \to 1^-}{Lt} \dfrac{1-1}{x-1}$

$= 0$

$RHD = \underset{x \to 1^+}{Lt} \dfrac{f(x) - f(1)}{x-1}$

$= \underset{x \to 1^+}{Lt} \dfrac{2x-1-1}{x-1}$

$= \dfrac{2x-2}{x-1}$

$= \dfrac{2(x-1)}{x-1} = 2$

$\therefore LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x = 1$

6) Show that $f(x) = |x-1| + |x-2|$ has no derivative at $x = 1, 2$.

Sol: Given $f(x) = |x-1| + |x-2|$

Derivative at $x=1$

$x<1,\ x<2$

If $x<1 \Rightarrow |x-1| = -(x-1)$

If $x<2 \Rightarrow |x-2| = -(x-2)$

$f(x) = -x+1-x+2$

$= 3-2x$

$x>1,\ x<2$

If $x>1 \Rightarrow |x-1| = x-1$

If $x<2 \Rightarrow |x-2| = -(x-2)$

$f(x) = x-1-x+2$

$= 1$

Also, $f(1) = |1-1| + |1-2|$

$= 1$

$$f(x) = \begin{cases} 3-2x, & x<1 \\ 1, & x>1 \\ 1, & x=1 \end{cases}$$

$$LHD = \underset{x \to 1^-}{Lt} \frac{f(x) - f(c)}{x-c}$$

$$= \underset{x \to 1^-}{Lt} \frac{3-2x-1}{x-1}$$

$$= \frac{-2x+2}{x-1}$$

$$= \frac{-2(x-1)}{x-1}$$

$$= -2$$

$$RHD = \underset{x \to 1^+}{Lt} \frac{f(x) - f(1)}{x-1}$$

$$= \underset{x \to 1^+}{Lt} \frac{1-1}{x-1}$$

$$= 0$$

$LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x=1$

Derivative at $x=2$

$x>1,\ x<2$

If $x>1 \Rightarrow |x-1| = x-1$

If $x<2 \Rightarrow |x-2| = -(x-2)$

$f(x) = x-1-x+2$

$= 1$

$x>1,\ x>2$

If $x>1 \Rightarrow |x-1| = x-1$

If $x>2 \Rightarrow |x-2| = x-2$

$f(x) = x-1+x-2$

$= 2x-3$

Also, $f(2) = |2-1| + |2-2|$

$= 1$

$$f(x) = \begin{cases} 1 & , x < 2 \\ 2x-3 & , x > 2 \\ 1 & , x = 2 \end{cases}$$

$LHD = \underset{x \to 2^-}{Lt} \frac{f(x) - f(2)}{x-2}$

$= \underset{x \to 2^-}{Lt} \frac{1-1}{x-2}$

$= 0$

$RHD = \underset{x \to 2^+}{Lt} \frac{f(x) - f(2)}{x-2}$

$= \underset{x \to 2^+}{Lt} \frac{2x-3-1}{x-2}$

$= \underset{x \to 2^+}{Lt} \frac{2x-4}{x-2}$

$= \underset{x \to 2^+}{Lt} \frac{2(x-2)}{x-2}$

$= 2$

$LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x = 2$

7) prove that $f(x) = x \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(x) = 0$ is not derivative at $x = 0$

Sol: Given $f(x) = x \sin\left(\frac{1}{x}\right)$

$f(x) = 0$

$LHD = \underset{x \to 0^-}{Lt} \frac{f(x) - f(0)}{x-0}$

$= \underset{x \to 0^-}{Lt} \frac{x \sin\left(\frac{1}{x}\right) - 0}{x-0}$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$= \underset{h \to 0}{Lt} \sin\left(\frac{-1}{h}\right)$

$= -\sin\left(\frac{1}{h}\right)$

$= -$infinite value

$RHD = \underset{x \to 0^+}{Lt} \frac{f(x) - f(0)}{x-0}$

$= \underset{x \to 0^+}{Lt} \frac{x \sin\left(\frac{1}{x}\right) - 0}{x-0}$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$= \underset{h \to 0}{Lt} \sin\left(\frac{1}{h}\right)$

$=$ infinite value

$LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x = 0$

8) P.T $f(x) = x \cos\left(\frac{1}{x}\right)$, if $x \neq 0$ and $f(x) = 0$ is derivative at $x = 0$

Sol: Given $f(x) = x \cos \frac{1}{x}$

$f(0) = 0.$

$LHD = \underset{x \to 0^-}{Lt} \frac{f(x)-f(0)}{x-0}$

$= \underset{x \to 0^-}{Lt} \frac{x\cos\left(\frac{1}{x}\right)-0}{x-0}$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$= \underset{h \to 0^-}{Lt} \cos\left(\frac{1}{h}\right)$

$= \cos\left(\frac{1}{h}\right)$

$=$ infinite value

$RHD = \underset{x \to 0^+}{Lt} \frac{f(x)-f(0)}{x-0}$

$= \underset{x \to 0^+}{Lt} \frac{x\cos\left(\frac{1}{x}\right)-0}{x-0}$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$= \underset{h \to 0^+}{Lt} \cos\left(\frac{1}{h}\right)$

$=$ inifinite value

$\therefore LHD = RHD$

$\therefore f(x)$ has derivative at $x=0$

Q) prove that $f(x) = x \, Tan^{-1}\left(\frac{1}{x}\right)$ if $x \neq 0$, and $f(x) = 0$ is not derivative at $x=0$.

Sol: Given $f(x) = x\,Tan^{-1}\left(\frac{1}{x}\right)$

$f(0) = 0$

$LHD = \underset{x \to 0^-}{Lt} \frac{f(x)-f(0)}{x-0}$

$= \underset{x \to 0^-}{Lt} \frac{x\,Tan^{-1}\left(\frac{1}{x}\right)-0}{x-0}$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$= \underset{h \to 0}{Lt} Tan^{-1}\left(-\frac{1}{h}\right)$

$= \underset{h \to 0}{Lt} -Tan^{-1}\left(\frac{1}{h}\right)$

$= -Tan^{-1}\left(\frac{1}{0}\right)$

$= -Tan^{-1}(\alpha)$

$= \frac{-\pi}{2}$

$RHD = \underset{x \to 0^+}{Lt} \frac{f(x)-f(0)}{x-0}$

$= \underset{x \to 0^+}{Lt} \frac{x\tan^{-1}\left(\frac{1}{x}\right)-0}{x-0}$

$x \to 0^+$ means $x = 0-h$ as $h \to 0$

$= \underset{h \to 0}{Lt} Tan^{-1}\left(\frac{1}{h}\right)$

$= Tan^{-1}\left(\frac{1}{0}\right)$

$= Tan^{-1}(\alpha)$

$= \frac{\pi}{2}$

$\therefore LHD \neq RHD$

$\therefore f(x)$ has no derivative at $x=0$.

10) S.T $f(x) = \frac{x}{1+e^{1/x}}$, if $x \neq 0$, and $f(x)=0$, if $x \neq 0$, and $f(x)=0$, if $x=0$.

Sol: Given $f(x) = \frac{x}{1+e^{1/x}}$ and

$f(0)=0$

$$LHD = \underset{x\to 0^-}{Lt} \frac{\left(\frac{x}{1+e^{1/x}}\right) - f(0)}{x-0}$$

$$= \underset{x\to 0^-}{Lt} \frac{\frac{x}{1+e^{1/x}} - 0}{x}$$

$$= \underset{x\to 0}{Lt} \frac{1}{1+e^{1/x}}$$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$$= \underset{h\to 0}{Lt} \frac{1}{1+e^{-1/h}}$$

$$= \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$$RHD = \underset{x\to 0^+}{Lt} \frac{f(x)-f(0)}{x-0}$$

$$= \underset{x\to 0^+}{Lt} \frac{\left(\frac{x}{1+e^{1/x}}\right) - f(0)}{x-0}$$

$$= \underset{x\to 0^+}{Lt} \frac{1}{1+e^{1/x}}$$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$$= \underset{h\to 0}{Lt} \frac{1}{1+e^{1/h}}$$

$$= \underset{h\to 0}{Lt} \frac{1}{1+\frac{1}{e^{-1/h}}}$$

$$= \underset{h\to 0}{Lt} \frac{e^{-1/h}}{e^{-1/h}+1} = \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \frac{0}{0+1} = 0$$

$\therefore$ LHD $\neq$ RHD

$\therefore$ $f(x)$ has no derivative at $x=0$

Theorem:

Statement: If 'f' is derivative at a point 'c' then it is continuous at c. Is converse true?

Proof: suppose 'f' is derivative at 'c'.

ie, $\underset{x\to c}{Lt} \frac{f(x)-f(c)}{x-c} = f'(c)$ exists

Now, We prove that f is continuous at c

ie $\underset{x\to c}{Lt} f(x) = f(c)$

$$\underset{x\to c}{Lt} f(x) - f(c) \Rightarrow \underset{x\to c}{Lt} \frac{f(x)-f(c)}{x-c}(x-c)$$

$$\Rightarrow \left[\underset{x\to c}{Lt} \frac{f(x)-f(c)}{x-c}\right]\left[\underset{x\to c}{Lt} (x-c)\right]$$

$\Rightarrow f'(c) \times 0 = 0$

$\Rightarrow \underset{x \to c}{Lt} f(x) - f(c) = 0$

$\Rightarrow \underset{x \to c}{Lt} f(x) = f(c)$

'f' is continuous at c

Converse part:

consider $f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \end{cases}$

$LHD = \underset{x \to 0}{Lt} (-x)$

$= 0$

$RHD = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} x$

$= 0$

Also $f(0) = 0$

$\therefore \underset{x \to 0}{Lt} f(x) = f(0) = f(c)$

$f(c)$ is continuous at $x = 0$

Derivative at $x=0$:

$LHD = \underset{x \to 0^-}{Lt} \frac{f(x) - f(0)}{x - 0}$

$= \underset{x \to 0^-}{Lt} \frac{-x - 0}{x}$

$= -1$

$RHD = \underset{x \to 0^+}{Lt} \frac{f(x) - f(0)}{x - 0}$

$= \underset{x \to 0^+}{Lt} \frac{x - 0}{x - 0}$

$= 1$

$LHD \neq RHD$

$\therefore f(x)$ is not derivative at $x = 0$

$\therefore$ converse part is not true.

2) Rolle's theorem :

Statement : If a function 'f' is defined on $[a,b]$ is i) continuous on $[a,b]$ ii) Derivable on $[a,b]$ iii) $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$

proof : suppose 'f' is

i) continuous on $[a,b]$

ii) Derivable on $[a,b]$

iii) $f(a) = f(b)$

Now, We prove that there exist $c \in (a,b)$ such that $f'(c) = 0$

Since 'f' is continuous on $[a,b]$, 'f' is bounded on $[a,b]$.

'f' attains its bounce on $[a,b]$

let $\sup f = M$ and $\operatorname{Inf} f = m$

then there exists $c \in (a,b) \Rightarrow f(c) = M$

and $d \in (a,b) \Rightarrow f(d) = m$

case (i) : $\sup f = \operatorname{Inf} f$

i.e, $M = m$

$\Rightarrow$ f is continuous on $[a,b]$

Let $f(x) = k$ (constant)

then $f'(x) = 0$ (or) $f'(c) = 0$

case (ii) : $\sup. f \neq \operatorname{Inf} f$

ie $M \neq m$

$f(a) \neq M$ and $f(b) \neq m$

$f(a) \neq f(c) \Rightarrow a \neq c$

$f(b) \neq f(c) \Rightarrow b \neq c$

$\therefore c \in (a,b)$

$\therefore$ Since 'f' is derivable on (a,b) 'f' is derivable at 'c'.

$\therefore$ LHD and RHD exists at 'c'.

$$LHD = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \qquad RHD = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

$= \underset{x \to c}{Lt} \dfrac{f(c-h) - f(c)}{(c-h) - c}$ $\quad = \underset{x \to c^+}{Lt} \dfrac{f(c+h) - f(c)}{c+h-c}$

$= \underset{x \to c^-}{Lt} \dfrac{f(c-h) - f(c)}{-h} \longrightarrow ①$ $\quad = \underset{x \to c^+}{Lt} \dfrac{f(c+h) - f(c)}{h} \longrightarrow ②$

Since $f(c) = M = \sup f$

$f(c)$ is upper bound of f on $[a,b]$

$f(c-h) \le f(c)$, $f(c+h) \le f(c)$

$\Rightarrow f(c-h) - f(c) \le 0$, $f(c+h) - f(c) \le 0$

substituting these values in equ ①

then LHD ≥ 0 and RHD ≤ 0

$\Rightarrow f'(c) = 0$

Hence proved

Problems :-

Verify Robles theorem for the following :

1) $f(x) = x^2 - 6x + 8$ in $[2,4]$

Sol: Given $f(x) = x^2 - 6x + 8$

'f' is polynomial function of degree '2'.

f is continuous on $[2,4]$

f is derivable on $(2,4)$

$f(2) = 2^2 - 6(2) + 8$

$= 4 - 12 + 8$

$= 12 - 12$

$= 0$

$f(4) = 4^2 - 6(4) + 8$

$= 16 - 24 + 8$

$= 24 - 24$

$= 0$

$\therefore f(2) = f(4)$

$\therefore$ Rolles theorem is applicable for 'f'

$f(x) = x^2 - 6x + 8$

$f'(x) = 2x - 6$

$f'(c) = 2c - 6$

Write $f'(c) = 0$

ie $2c - 6 = 0$

$2c = 6$

$\boxed{c = 3}$

$\therefore c \in (2,4)$

$\therefore$ Rolle's theorem is verified.

2) $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1,3]$

Sol: Given $f(x) = x^3 - 6x^2 + 11x - 6$

'f' is polynomial function of degree '3'.

f is continuous on $[1,3]$

f is derivable on $(1,3)$

$f(1) = 1^3 - 6(1)^2 + 11(1) - 6$

$= 1 - 6 + 11 - 6$

$= 12 - 12$

$= 0$

$f(3) = 3^3 - 6(3)^2 + 11(3) - 6$

$= 27 - 54 + 33 - 6$

$= 60 - 60$

$= 0$

$\therefore f(1) = f(3)$

$\therefore$ Rolle's theorem is applicable for 'f'.

$f(x) = x^3 - 6x^2 + 11x - 6$

$$f'(x) = 3x^2 - 12x + 11$$

$$f'(c) = 3c^2 - 12c + 11$$

Write $f'(c) = 0$

i.e, $3c^2 - 12c + 11 = 0$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{144 - 4(3)(11)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{144 - 132}}{6}$$

$$= \frac{12 \pm \sqrt{12}}{6}$$

$$= \frac{12 \pm 2\sqrt{3}}{6}$$

$$c = \frac{2(6 \pm \sqrt{3})}{6}$$

$$c = \frac{6 \pm \sqrt{3}}{3}$$

$\therefore c \in (1, 3)$

$\therefore$ Rolle's theorem is verified.

3) $f(x) = x^3 - 4x$ in $[-2, 2]$

Sol: Given $f(x) = x^3 - 4x$

'f' is polynomial function of degree '3'.

f is continuous on $[-2, 2]$

f is derivable on $(-2, 2)$

$$f(-2) = (-2)^3 - 4(-2)$$

$$= -8 + 8 = 0$$

$f(2) = (2)^3 - 4(2)$

$= 8 - 8$

$= 0$

$\therefore f(-2) = f(2)$

$\therefore$ Rolle's theorem is applicable for 'f'.

$f(x) = x^3 - 4x$

$f'(x) = 3x^2 - 4$

$f'(c) = 3c^2 - 4$

Write $f'(c) = 0$

ie, $3c^2 - 4 = 0$

$3c^2 = 4$

$c^2 = \frac{4}{3}$

$c = \sqrt{\frac{4}{3}}$

$c = \frac{2}{1.732} = 1.1547$

$\therefore c \in (-2, 2)$

$\therefore$ Rolle's theorem is verified.

4) $f(x) = 8x - x^2$ in $[2, 6]$

Sol: Given $f(x) = 8x - x^2$

f is polynomial function of degree '2'

f is continuous on $[2, 6]$

f is derivable on $(2, 6)$

$f(2) = 8(2) - (2)^2$

$= 16 - 4 = 12$

$f(6) = 8(6) - (6)^2$

$= 48 - 36$

$= 12$

$\therefore f(2) = f(6)$

$\therefore$ Roll's theorem is applicable for 'f'

$f(x) = 8x - x^2$

$f'(x) = 8 - 2x$

$f'(c) = 8-2c$

Write $f'(c)=0$

ie $8-2c=0$

$2c=8$

$c=8/2$

$c=4$

$\therefore c\in(2,6)$

$\therefore$ Rolle's theorem is verified.

5) verify Rolle's theorem for $f(x)=\sin x$ in $[0,2\pi]$.

<u>Sol</u>: Given $f(x)=\sin x$

'f' is trigonometric function

f is continuous on $[0,2\pi]$

f is derivable on $(0,2\pi)$

$f(0)=\sin(0)$

$=0$

$f(2\pi)=\sin(2\pi)$

$=0$

$\therefore f(0)=f(2\pi)$

$\therefore$ Rolle's theorem is applicable for 'f'

$f(x)=\sin x$

$f'(x)=\cos x$

$f'(c)=\cos c$

Write $f'(c)=0$

ie $\cos c=0$

$c=\frac{\pi}{2},\frac{3\pi}{2}$

$\therefore c\in(0,2\pi)$

$\therefore$ Rolle's theorem is verified.

6) $f(x) = \cos x$ in $[\pi, 3\pi]$

Sol: Given $f(x) = \cos x$

'f' is trigonometric function

f is continuous on $[\pi, 3\pi]$

f is derivable on $(\pi, 3\pi)$

$f(\pi) = \cos(\pi)$

$= -1$

$f(3\pi) = \cos(3\pi)$

$= -1$

$\therefore f(\pi) = f(3\pi)$

$\therefore$ Rolle's theorem is applicable for 'f'

$f(x) = \cos x$

$f'(x) = -\sin x$

$f'(c) = -\sin c$

Write $f'(c) = 0$

ie $-\sin c = 0$

$c = 2\pi$

$\therefore c \in (\pi, 3\pi)$

$\therefore$ Rolle's theorem is verified.

7) verify Rolle's theorem for $f(x) = (x-a)(x-b)$ in $[a,b]$

Sol: Given 'f' is polynomial function of order '2'

'f' is continuous on $[a,b]$

'f' is derivable on (a,b)

Now $f(a) = (a-a)(a-b)$

$= 0$

$f(b) = (b-a)(b-b)$

$= 0$

$\therefore f(a) = f(b) = 0$

$\therefore$ Rolle's theorem is applicable for 'f'.

$f(x) = (x-a)(x-b)$

$f'(x) = (x-a)+(x-b)$

$f'(c) = (c-a)+(c-b)$

$= 2c-a-b$

Write $f'(c)=0$

ie, $2c-a-b=0$

$c = \frac{a+b}{2}$

$c \in (a,b)$

$\therefore$ Rolle's theorem is verified.

8) verify Rolle's theorem for $f(x)=\sqrt{1-x^2}$ in $[-1,1]$

Sol: Given 'f' is polynomial function of order '2'.

'f' is continuous on $[-1,1]$

'f' is derivable on $(-1,1)$

Now, $f(-1) = \sqrt{1-(-1)^2}$ $\quad f(1) = \sqrt{1-1^2}$

$= 0$ $\quad = 0$

$\therefore f(-1) = f(1) = 0$

$\therefore$ Rolle's theorem is applicable for 'f'

$f(x) = \sqrt{1-x^2}$

$f'(x) = \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)$

$f'(x) = \frac{-x}{\sqrt{1-x^2}}$

$f'(c) = \frac{-c}{\sqrt{1-c^2}}$

Write $f'(c) = 0$

$\Rightarrow \frac{-c}{\sqrt{1-c^2}} = 0$

$\Rightarrow c = 0$

$c \in (-1, 1)$

$\therefore$ Rolle's theorem is verified.

9) $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$

Sol: Given 'f' is polynomial function of order '2'.

'f' is continuous on $[0, 2]$

'f' is derivable on $(0, 2)$

Now, $f(0) = 2 + (0-1)^{2/3}$ $\qquad$ $f(2) = 2 + (2-1)^{2/3}$

$= 2 + 1$ $\qquad$ $= 2 + 1$

$= 3$ $\qquad$ $= 3$

$\therefore f(0) = f(2)$

$\therefore$ Rolle's theorem is applicable for 'f'.

$$f(x) = 2 + (x-1)^{2/3}$$

$$f'(x) = \frac{2}{3}(x-1)^{2/3 - 1}$$

$$f'(x) = \frac{2}{3}(x-1)^{-1/3}$$

$$f'(c) = \frac{2}{3}(c-1)^{-1/3}$$

Write $f'(c) = 0$

$\Rightarrow \frac{2}{3}(c-1)^{-1/3} = 0$

$\Rightarrow c - 1 = 0$

$\Rightarrow c = 1$

$c \in (0, 2)$

$\therefore$ Rolle's theorem is verified.

Imp

10) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$

Sol: Given 'f' is polynomial function of order '2'.

'f' is continuous on $[-\sqrt{2}, \sqrt{2}]$

'f' is continuous on $(-\sqrt{2}, \sqrt{2})$

$$f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2$$
$$= 2(-2\sqrt{2}) + 2 + 4\sqrt{2} - 2$$
$$= 0$$

$$f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2$$
$$= 2(2\sqrt{2}) + 2 - 4\sqrt{2} - 2$$
$$= 0$$

$\cancel{f(\sqrt{2})}$

$$f(-\sqrt{2}) = f(\sqrt{2})$$

$\therefore$ Rolle's theorem is applicable for 'f'.

$$f(x) = 2x^3 + x^2 - 4x - 2$$
$$f'(x) = 6x^2 + 2x - 4 - 0$$
$$f'(x) = 6x^2 + 2x - 4$$
$$f'(c) = 6c^2 + 2c - 4$$

Write $f'(c) = 0$

$$\Rightarrow 6c^2 + 2c - 4 = 0$$
$$\Rightarrow 3c^2 + c - 2 = 0$$
$$\Rightarrow 3c^2 + 3c - 2c - 2 = 0$$
$$\Rightarrow 3c(c+1) - 2(c+1) = 0$$
$$\Rightarrow (c+1)(3c-2) = 0$$
$$\Rightarrow c = -1 \text{ (or) } c = \frac{2}{3}$$

$$\therefore c \in (-\sqrt{2}, \sqrt{2})$$

$\therefore$ Rolle's theorem is verified.

Lagrange's theorem:

Statement: A Function 'f' is defined on $[a,b]$

i) continuous on $[a,b]$

ii) Derivable on (a,b) then there exist atleast one real number $c\in(a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

proof: Suppose 'f' is continuous on $[a,b]$ and derivable on (a,b)

Now, We prove that there exists $c\in(a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Let $\phi(x) = f(x) + A(x) \rightarrow ①$

such that $\phi(a) = \phi(b)$

now $\phi(a) = f(a) + A.a$

$\phi(b) = f(b) + A.b$

$\phi(a) = \phi(b)$

$f(a) + A.a = f(b) + A.b$

$A.a - A.b = f(b) - f(a)$

$A(a-b) = f(b) - f(a)$

$$A = \frac{f(b) - f(a)}{a-b} \rightarrow ②$$

Since $f(x)$ is continuous and Ax is continuous on $[a,b]$

$\therefore \phi(x)$ is continuous on $[a,b]$

$\Rightarrow$ Since $f(x)$ is derivable on (a,b) and Ax derivable on (a,b)

$\therefore \phi(x)$ is derivable on (a,b)

Rolle's theorem is applied for $\phi(x)$

$\therefore$ There exists $c\in(a,b)$ such that $\phi'(c) = 0$

$\phi(x) = f(x) + Ax$

$\phi'(x) = f'(x) + A$

$\phi'(c) = f'(c) + A.$

$$0 = f'(c) + A$$

$$f'(c) = -A \Rightarrow -\left[\frac{f(b) - f(a)}{a-b}\right]$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

$\therefore$ Hence proved.

1) verify lagrange's theorem for $f(x) = x(x-1)(x-2)$ on $[0, \frac{1}{2}]$

Sol: Given $f(x) = x(x-1)(x-2)$ ~~on~~

$$= x(x^2 - 3x + 2)$$

$$= x^3 - 3x^2 + 2x$$

$f(x)$ is polynomial Function of degree 3

f is continuous on $[0, \frac{1}{2}]$

f is derivable on $(0, \frac{1}{2})$

$$f(x) = x^3 - 3x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)$$

$$= \frac{1}{8} - \frac{3}{4} + \frac{2}{2}$$

$$= \frac{1 - 6 + 8}{8}$$

$$= \frac{3}{8}$$

$$f(0) = 0$$

By Lagranges theorem $\exists\, c \in (0, \frac{1}{2})$ such that

$$f'(c) = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

$$= \frac{3/8}{1/2}$$

$$= \frac{3}{4}$$

$$3c^2 - 6c + 2 = \frac{3}{4}$$

$$12c^2 - 24c + 8 = 3$$

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$= \cancel{\frac{1 \pm \sqrt{336}}{2}}$$

$$= \frac{1 \pm \sqrt{336}}{24}$$

$c = 1 + 0.76$, $c = 1 - 0.76$

$= 1.76$, $= 0.244$

$c = 1.76 \notin (0, \frac{1}{2})$

$c = 0.244 \in (0, \frac{1}{2})$

$\therefore$ Lagranges theorem verified.

2) $f(x) = (x-1)(x-2)(x-3)$ on $[0, 4]$

Sol: Given $f(x) = (x-1)(x-2)(x-3)$

$$= (x-1)(x^2 - 5x + 6)$$

$$= x^3 - 5x^2 + 6x - x^2 + 5x - 6$$

$$= x^3 - 6x^2 + 11x - 6$$

$f(x)$ is polynomial Function of degree '3'.

f is continuous on $[0,4]$
f is derivable on $(0,4)$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$

$f(0) = -6$

$f(4) = (4)^3 - 6(4)^2 + 11(4) - 6$

$= 64 - 96 + 44 - 6$

$= 6$

By Lagranges theorem $\exists\, c \in (0,4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$3c^2 - 12c + 11 = \frac{6+6}{4} = \frac{12}{4} = \frac{6^3}{2}$$

$$3c^2 - 12c + 8 = 0$$

$$c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$$= \frac{12 \pm 6.9}{6}$$

$$c = \frac{12 + 6.9}{6} \qquad c = \frac{12 - 6.9}{6}$$

$$= \frac{18.9}{6} \qquad = \frac{5.1}{6}$$

$$= 3.15 \qquad = 0.85$$

$c \in (0, 4)$

Lagranges theorem verified.

3) $f(x) = x^2+1$ on $[1,2]$

Sol: Given $f(x) = x^2+1$

$f(x)$ is polynomial function of degree '2'

f is continuous on $[1,2]$

f is derivable on $(1,2)$

$f(x) = x^2+1$

$f'(x) = 2x$

$f'(c) = 2c$

$f(1) = 1^2+1$

$= 2$

$f(2) = 2^2+1$

$= 4+1$

$= 5$

By Lagranges theorem $\exists$ $c \in (1,2)$ such that

$$f'(c) = \frac{f(2)-f(1)}{2-1}$$

$$2c = \frac{5-2}{1}$$

$$c = \frac{3}{2} = 1.5 \in (1,2)$$

Lagranges theorem is verified.

4) $f(x) = x^2-2x+3$ on $\left[1, \frac{3}{2}\right]$

Sol: Given $f(x) = x^2-2x+3$

$f(x)$ is polynomial function of degree '2'

f is continuous on $\left[1, \frac{3}{2}\right]$

f is derivable on $\left(1, \frac{3}{2}\right)$

$f(x) = x^2-2x+3$

$f'(x) = 2x-2$

$f'(c) = 2c-2$

$f(1) = 1^2 - 2(1) + 3 = 2$ $\quad f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right) + 3$

$= \frac{9}{4} - 3 + 3$

$= 9/4$

By Lagranges theorem $\exists\, c \in \left(1, \frac{3}{2}\right)$ such that

$$f'(c) = \frac{f\left(\frac{3}{2}\right) - f(1)}{\frac{3}{2} - 1}$$

$$2c - 2 = \frac{\frac{9}{4} - 2}{\frac{3-2}{2}} \Rightarrow \frac{\frac{9-8}{\cancel{4}\,2}}{1/\cancel{2}}$$

$$2(c-1) = \frac{1}{2}$$

$$c - 1 = \frac{1}{4}$$

$$c = \frac{1}{4} + 1 = \frac{5}{4} = 1.2$$

$$c \in \left(1, \frac{3}{2}\right)$$

$\therefore$ Lagranges theorem verified.

5) $f(x) = x^3$ on $\left[1, \frac{3}{2}\right]$

Sol: Given $f(x) = x^3$

$f(x)$ is polynomial function of degree '3'

f is continuous on $\left[1, \frac{3}{2}\right]$

f is derivable on $\left(1, \frac{3}{2}\right)$

$f(x) = x^3$

$f(x) = 3x^2$

$f'(c) = 3c^2$

$f(1) = 1^3 = 1$

$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

By Lagranges theorem $\exists\, c \in \left[1, \frac{3}{2}\right]$ such that

$$f'(c) = \frac{f(\frac{3}{2}) - f(1)}{\frac{3}{2} - 1}$$

$$3c^2 = \frac{\frac{27}{8} - 1}{\frac{1}{2}}$$

$$c^2 = \frac{19}{4 \times 3} = \frac{19}{12}$$

$$c = \sqrt{\frac{19}{12}} = 1.2583$$

$$c \in (1, \frac{1}{3})$$

Lagrangers theorem is verified.

6) $f(x) = 4x^2$ on $[1, 2]$

Sol: Given $f(x) = 4x^2$

$f(x)$ is polynomial function of degree '2'

f is continuous on $[1, 2]$

f is derivable on $(1, 2)$

$$f(x) = 4x^2$$

$$f'(x) = 8x$$

$$f'(c) = 8c$$

$f(1) = 4$ $\quad$ $f(2) = 4(2)^2 = 16$

By Lagrange's theorem $\exists\, c \in (1, 2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$8c = \frac{16 - 4}{1}$$

$$8c = 12$$

$$c = \frac{12}{8}$$

$$c = \frac{3}{2} = 1.5$$

$$c \in (1, 2)$$

Lagrange's theorem is verified.

7) P.T $x > \log(1+x) > \frac{x}{x+1}$ if $f(x) = \log(1+x)$, $x>0$

Sol: Given $f(x) = \cancel{x} \log(1+x)$, $x>0$ $\forall x \in (0,t)$ $t>0$

f is continuous on $[0,t]$

f is derivable on $(0,t)$

By lagrange's theorem $\exists\, c \in (0,1)$ such that

$$f'(c) = \frac{f(t)-f(0)}{t-0} \rightarrow (1)$$

$$f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(c) = \frac{1}{1+c}$$

$$f(t) = \log(1+t),\ f(0) = \log(1+0) = \log 1$$

$$f'(c) = \frac{\log(1+t) - \log 1}{t} \quad (\because \log 1 = 0)$$

$$\frac{1}{1+c} = \frac{\log(1+t)}{t} \rightarrow (2)$$

Since $c \in (0,t) \Rightarrow 0<c<t$

Adding 1 $\Rightarrow 1+0 < 1+c < 1+t$

$$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+t}$$

$$\Rightarrow 1 > \frac{\log(1+t)}{t} > \frac{1}{1+t} \quad [\because \text{From } (2)]$$

$$\Rightarrow t > \log(1+t) > \frac{t}{1+t},\ t>0$$

$$\Rightarrow x > \log(1+x) > \frac{x}{1+x},\ x>0.$$

8) P.T ~~f(x) =~~ $1+x < e^x < 1+x \cdot e^x$ if $f(x) = e^x, x>0$

Sol: Given that $f(x) = e^x, x>0 \; \forall \; x \in [0,t], t>0$

f is continuous on $[0,t]$

f is ~~conti~~ derivable on $(0,t)$

By Lagranges theorem $\exists \; c \in (0,t)$ such that

$$f'(c) = \frac{f(t)-f(0)}{t-0} \rightarrow ①$$

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f'(c) = e^c, \; f(t) = e^t, \; f(0) = e^0 = 1$$

put these values in equ ①

$$e^c = \frac{e^t - 1}{t} \rightarrow ②$$

since $c \in (0,t)$

$\Rightarrow 0 < c < t$

$\Rightarrow e^0 < e^c < e^t$

$\Rightarrow 1 < \frac{e^t - 1}{t} < e^t$ [$\because$ From equ ②]

$\Rightarrow t < e^t - 1 < te^t$ > Adding 1

$\Rightarrow 1+t < e^t < te^t + 1, t>0$

Hence proved.

Imp A

9) Using Lagranges theorem show that $\frac{V-U}{1+V^2} < \tan^{-1} V - \tan^{-1} U < \frac{U-V}{1+U^2}$

if $0 < U < V$ to deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$.

Sol: Given $f(x) = \tan^{-1} x$

f is continuous on $[U,V]$

f is derivable on (U,V)

By Lagranges theorem $\exists \; c \in (U,V)$ such that

$$f(c) = \frac{f(V) - f(U)}{V-U} \rightarrow ①$$

$$f(x) = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(c) = \frac{1}{1+c^2}$$

$f(v) = Tan^{-1}v$, $f(u) = Tan^{-1}u$

put these values in equ 1

$$\frac{1}{1+c^2} = \frac{Tan^{-1}v - Tan^{-1}u}{v-u}$$

since $c \in (u,v)$

$\Rightarrow u < c < v$

$\Rightarrow u^2 < c^2 < v^2$

Adding 1

$\Rightarrow 1+u^2 < 1+c^2 < 1+v^2$

$$\Rightarrow \frac{1}{1+u^2} > \frac{1}{1+c^2} > \frac{1}{1+v^2}$$

$$\Rightarrow \frac{1}{1+u^2} > \frac{Tan^{-1}v - Tan^{-1}u}{v-u} > \frac{1}{1+v^2}$$

$$\Rightarrow \frac{v-u}{1+u^2} > Tan^{-1}v - Tan^{-1}u > \frac{v-u}{1+v^2}$$

$$\Rightarrow \frac{v-u}{1+u^2} < Tan^{-1}v - Tan^{-1}u < \frac{v-u}{1+v^2} \longrightarrow 3$$

Adding $Tan^{-1}u$ in 3

$$\Rightarrow Tan^{-1}u + \frac{v-u}{1+v^2} < Tan^{-1}v < Tan^{-1}u + \frac{v-u}{1+u^2}$$

~~Adding~~ putting $v = 4/3$ ~~(or)~~ and $u = 1$

$$\Rightarrow Tan^{-1}(1) + \frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < Tan^{-1}(4/3) < Tan^{-1}(1) + \frac{\frac{4}{3}-1}{1+1}$$

$$\Rightarrow \frac{\pi}{4} + \frac{1}{25} < Tan^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$$

Hence proved.

10) S.T $1-\frac{a}{b} < \log\frac{b}{a} < \frac{b}{a}-1$.

Sol: Let $f(x) = \log x$

f is continuous on $[a,b]$

f is derivable on (a,b)

By lagrange's theorem $\exists\, c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \longrightarrow \text{equ}①$$

$$f(x) = \log x \Rightarrow f'(x) = 1/x \Rightarrow f'(c) = \frac{1}{c}$$

$$f(b) = \log b \Rightarrow f(a) = \log a$$

putting these values in Equ ①

$$\frac{1}{c} = \frac{\log b - \log a}{b-a}$$

$$\frac{1}{c} = \frac{\log(b/a)}{b-a}$$

Since $c \in (a,b)$

$$a < c < b$$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{1}{a} > \frac{\log(b/a)}{b-a} > \frac{1}{b}$$

$$\frac{b-a}{a} > \log(b/a) > \frac{b-a}{b}$$

$$\frac{b}{a}-1 > \log(b/a) > 1-\frac{a}{b}$$

$$1-\frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a}-1$$

11) S.T $\frac{x}{1+x^2} < \tan^{-1}x < x, \forall\, x > 0$ & deduce that $2 < \pi < 4$

Sol: $f(x) = \tan^{-1}x, \forall\, x > 0 \quad x \in [0,t]$

f is continuous on $(0,t)$

f is derivable on $(0,t)$.

By lagranges theorem $\exists\, c \in (0,t)$ such that

$$f'(c) = \frac{f(t)-f(0)}{t-0} \longrightarrow \text{equ } ①$$

$$f(x) = \text{Tan}^{-1}x$$

$$f'(x) = \frac{1}{1+x^2}, \quad f'(c) = \frac{1}{1+c^2}$$

$$f(t) = \text{Tan}^{-1}t, \quad f(0) = \text{Tan}^{-1}0 = 0$$

putting these values in equ ①

$$\frac{1}{1+c^2} = \frac{\text{Tan}^{-1}t}{t} \longrightarrow \text{equ } ②$$

since $c \in (0,t)$

$$0 < c < t$$

$$0 < c^2 < t^2$$

Adding 1 $\Rightarrow$ 1 to $< 1+c^2 < 1+t^2$

$$1 > \frac{1}{1+c^2} > \frac{1}{1+t^2}$$

$$1 > \frac{\text{Tan}^{-1}(t)}{t} > \frac{1}{1+t^2}$$

$$t > \text{Tan}^{-1}(t) > \frac{t}{1+t^2}$$

$$\frac{t}{1+t^2} < \text{Tan}^{-1}(t) < t, \; t > 0$$

$$\frac{x}{1+x^2} < \text{Tan}^{-1}x < x, \; x > 0$$

let $x = 1$

Then $\frac{1}{1+1^2} < Tan^{-1} < 1$

$\frac{1}{2} < Tan^{-1}\left(Tan \frac{\pi}{4}\right) < 1$

$\frac{1}{2} < \frac{\pi}{4} < 1$

$\frac{4}{2} < \pi < 4$ [$\because$ Multiply with '4']

$2 < \pi < 4$

12) P.T $0 < \frac{1}{x} \log\left[\frac{e^x - 1}{x}\right] < 1, \forall x > 0$

Sol: Let $f(x) = e^x, \forall x \in [0,t]$

f is continuous on $[0,t]$

f is derivable on $(0,t)$

By lagrange's theorem $\exists$ $c \in (0,t)$ such that

$$f'(c) = \frac{f(t) - f(0)}{t - 0} \rightarrow \text{equ } ①$$

$f(x) = e^x$, $f'(x) = e^x$, $f'(c) = e^c$

$f(t) = e^t$, $f(0) = e^0 = 1$

putting these values in equ ①

$$e^c = \frac{e^t - 1}{t} \rightarrow \text{equ } ②$$

since $c \in (0, t)$

$0 < c < t$

$e^0 < e^c < e^t$

$1 < e^c < e^t$

$1 < \frac{e^t - 1}{t} < e^t$ [$\because$ From ②]

Taking logarithm

$\log 1 < \log\left(\frac{e^t - 1}{t}\right) < \log e^t$

$\Rightarrow 0 < \log\left(\frac{e^t - 1}{t}\right) < t \log e$

$\Rightarrow \frac{0}{t} < \frac{1}{t}\log\left(\frac{e^t - 1}{t}\right) < 1$

$\Rightarrow 0 < \frac{1}{t}\log\left(\frac{e^t - 1}{t}\right) < 1, \ t > 0$

$\Rightarrow 0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1, \ x > 0$

∴ Hence proved.

13) P.T $\tan^{-1} v - \tan^{-1} u < v - u$, where $v > u > 0$.

Sol: Let $f(x) = \tan^{-1} x, \ \forall x > 0, \ x \in [u, v]$

'f' is continuous on $[u, v]$

F is derivable on (u, v)

By L.T $\exists$ $c \in (u, v)$ such that

$$f'(c) = \frac{f(v) - f(u)}{v - u}$$

$f(x) = \tan^{-1} x$, $f'(x) = \frac{1}{1+x^2}$, $f'(c) = \frac{1}{1+c^2}$

putting these values in ①

$$\frac{1}{1+c^2} = \frac{\tan^{-1} v - \tan^{-1} u}{v - u}$$

Since $c \in (u, v)$

$\Rightarrow v > u > 0$

$\Rightarrow v - u > 0$

$\Rightarrow 0 < v - u < 1$

$\Rightarrow 0 < \frac{1}{1+c^2} < 1$

$\Rightarrow 0 < \dfrac{\tan^{-1}v - \tan^{-1}u}{v-u} < 1$

$\Rightarrow \tan^{-1}v - \tan^{-1}u < v-u$

Hence proved

Cauchy's mean value theorem :

statement : If the Function f, g are defined on [a, b] are

i) continuous on [a, b]

ii) Derivable on (a, b)

iii) $g(x) \neq 0$, $\forall x \in (a,b)$ then $\exists$ atleast one real number $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

proof : Given that 'f' and 'g' are continuous on [a, b] and derivable on (a, b) and $g'(x) \neq 0$

now, we P.T $\exists$ $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

For this consider, $\phi(x) = f(x) + Ag(x)$, $\forall x \in (a,b)$

such that $\phi(a) = \phi(b)$

$\phi(a) = f(a) + Ag(a)$

$\phi(b) = f(b) + Ag(b)$

$\phi(a) = \phi(b)$

ie, $f(a) + Ag(a) = f(b) + Ag(b)$

$Ag(a) - Ag(b) = f(b) - f(a)$

$$-A = \frac{f(b)-f(a)}{g(a)-g(b)} \longrightarrow ①$$

Since, f, g are continuous on [a, b]

$\therefore \phi(x)$ is continuous on [a, b]

Since f, g are derivable on (a, b)

$\phi(x)$ is derivable on (a, b) and $\phi(a) = \phi(b)$

Rolles theorem is applicable for $\phi(x)$ then $\exists\ c \in (a,b)$ such that $\phi'(c) = 0$

$$\phi(x) = f(x) + A g(x), \quad \phi'(x) = f'(x) + A g'(x)$$

$$\phi'(c) = f'(c) + A g'(c)$$

$$f'(c) + A g'(c) = 0$$

$$-A = \frac{f'(c)}{g'(c)}$$

$$\frac{f'(c)}{g'(c)} = \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right]$$

$$= \frac{f(b) - f(a)}{g(b) - g(a)}$$

Hence proved.

1) verify cauchy's theorem for $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$ $0<a<$

Sol: Given $f(x) = \sqrt{x}$ then $f(x) = \frac{1}{2\sqrt{x}}$

$$f'(c) = \frac{1}{2\sqrt{x}}$$

$g(x) = \frac{1}{\sqrt{x}}$ then $g'(x) = -\frac{1}{2} x$

$$\Rightarrow -\frac{1}{2} x^{-3/2}$$

$$\Rightarrow \frac{-1}{2x\sqrt{x}} \text{ and}$$

$$g'(c) = -\frac{1}{2} c^{-3/2} = \frac{-1}{2c\sqrt{c}}$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

By cauchy's theorem

$$\Rightarrow \frac{\frac{1}{\sqrt{2c}}}{\frac{1}{\sqrt{2c}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\Rightarrow \quad c = \frac{\sqrt{ab}\,(\sqrt{b}-\sqrt{a})}{-c\,(\sqrt{b}-\sqrt{a})}$$

$$\Rightarrow \quad c = \sqrt{ab} \in (a,b)$$

2) verify C.T for $f(x) = x^2$, $g(x) = x^3$ in $[1, 2]$

<u>sol</u>: F, g are continuous on $[1, 2]$

f, g are derivable on $(1, 2)$

Given, $f(x) = x^2$, $f'(x) = 2x$, $f'(c) = 2c$

$g(x) = x^3$, $g'(c) = 3x^2$, $f'(c) = 3c^2$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{2c}{3c^2} = \frac{(2)^2-(1)^2}{(2)^3-1^3}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$c = \frac{14}{9}$$

$$c = 1.55$$

$$c \in (1, 2)$$

$\therefore$ cauchy's theorem is verified.

3) verify C.T $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a, b]$

<u>sol</u>: F, g are continuous on $[a, b]$ and derivable on (a, b)

given $f(x) = \frac{1}{x^2}$, $f(x) = -2x^{-2-1}$ $\qquad f'(c) = \frac{-2}{c^3}$

$= -2x^{-3} \Rightarrow \frac{-2}{x^3}$

$g(x) = \frac{1}{x}$, $g'(x) = -1 . x^{-1-1} = \frac{-1}{x^2}$

$g'(c) = \frac{-1}{c^2}$

From cauchy's theorem

$\Rightarrow \frac{-2/c^3}{-1/c^2} = \frac{\frac{-1}{b^2} - \frac{-1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$

$\Rightarrow \frac{2}{c} = \frac{a^2 - b^2/a^2 b^2}{(a-b)/ab}$

$\Rightarrow \frac{2}{c} = \frac{(a+b)(a-b)}{ab/a-b}$

$\Rightarrow \frac{2}{c} = \frac{(a+b)}{ab}$

$\Rightarrow c = \frac{2ab}{a+b}$

$\Rightarrow c \in (a,b)$

$\therefore$ cauchy's theorem is verified.

4) verify C.T for $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

Sol: F, g are continuous on $[a,b]$ and derivable on (a,b)

Given $f(x) = e^x$ $g(x) = e^{-x}$, $g'(c) = -e^{-c}$

By cauchy theorem $\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$

$\Rightarrow e^{2c} = \frac{e^{-(e^a - e^b)}}{e^a - e^b} \times e^a . e^b$

$\Rightarrow e^{2c} = e^{a+b}$

$\Rightarrow 2c = a+b$

$\Rightarrow c = \frac{a+b}{2}$

$c \in (a,b)$

cauchy's theorem is verified.